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AN APPROACH TO SEMI-MARKOV PROCESSES

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PREFACE

This paper reports on some of the mathematical results that the author obtained while seeking to refine the Markov chain models used in air ASW tactical analyses in numerous studies done at CNA. In an effort to take into account the waiting time in each state prior to transition, a non-Markov process was postulated. Subsequent investigation showed the process to be a reformulation of a semi-Markov process (c.f., references (a), (b), and (c)).

In the present case, the equations for the flow resemble a multi-dimensional renewal process. The behavior of the system is described by a probability density which characterizes the process at any time $t > 0$, given that the states of the process were defined at time $t = 0$. It is shown in the special steady state case that the probability distribution yields results which are equivalent to those previously given (references (a), (b), (c)).

The author wishes to thank Drs. J. Tyson, J. Bram, J. Kadane, and J. Howe for their valuable suggestions on certain aspects of the problem discussed in this paper.

INTRODUCTION

In dealing with a stationary Markov process with discrete state space $\{E_k\}_{k=1}^{N}$, we observe that the transition from state E_i to state E_j (for $i \neq j$) does not depend upon the amount of time the process has spent in state E_i. In fact, it follows as a consequence of the Markovian assumption that if at any time the process is observed to be in state E_i, and in addition, it is observed that the process next makes a transition to state E_j, the random variable T_{ij}, denoting the time the process spends in E_i before moving to E_j, is exponentially distributed. It follows that the probability of making the transition to E_j during the infinitesimal interval $[T_{ij}, T_{ij} + \Delta t)$ is just proportional to Δt, and does not depend on the amount of time, T_{ij}, already spent in E_i.

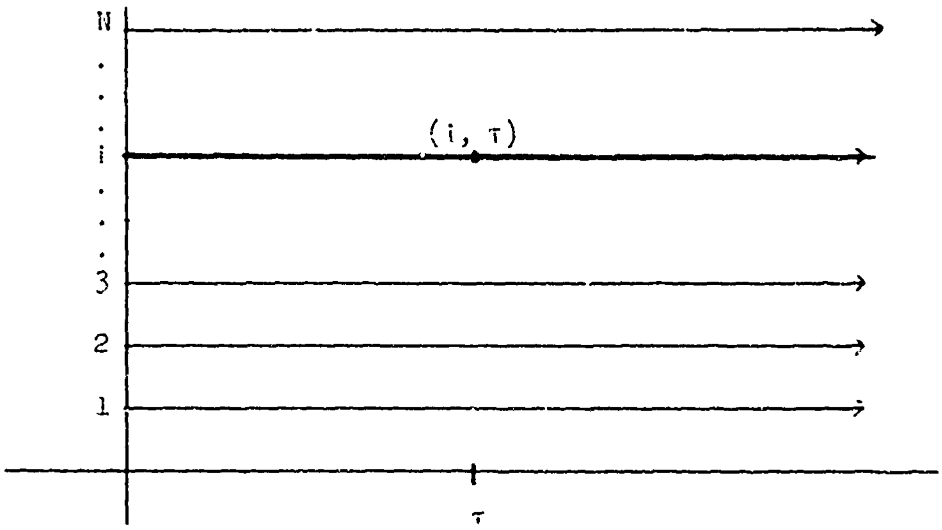
Systems arise though, in which the probability of transition from E_i to E_j does depend on the time already spent in E_i. That is, if it is known that the process has been in E_i for exactly the last τ units of time, the probability that a transition to E_j will occur in the next $\Delta\tau$ units of time depends not only on j and $\Delta\tau$ but on τ as well. We express this functionally as

$$w_{ij}(\tau)\Delta\tau = \text{Prob}\left\{\begin{matrix}\text{transition from } E_i \text{ to } E_j \text{ during} \\ \text{the time interval } [\tau, \tau + \Delta\tau)\end{matrix}\,\middle|\,\begin{matrix}\text{process has been in} \\ E_i \text{ throughout } [0, \tau)\end{matrix}\right\} \quad (1)$$

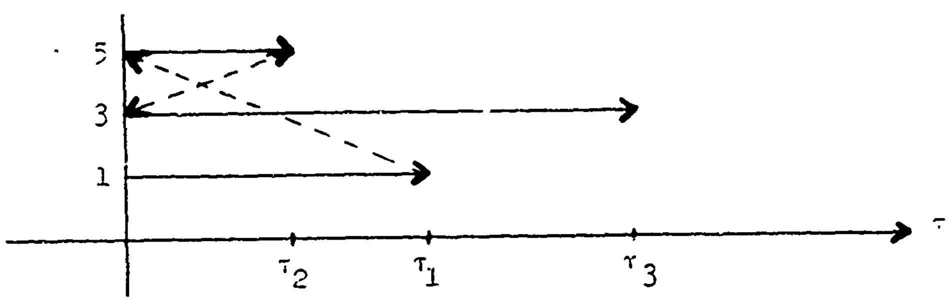
With a knowledge of these functions for all i,j plus a description of the initial states (at time t = 0), we can determine explicitly the condition of the process at any time t.

DEFINITION OF THE STATES

In order to make our process Markovian, we enlarge our state space. In essence we consider the time spent in state as part of the characterization of the state. More formally, we proceed as follows. Let E_1, E_2, ..., E_N denote the finite number of states of our original problem. We call these states elementary from now on. Now define a new set of states as follows. Each state is characterized by an ordered pair (i, τ) where i is an integer $1 \le i \le N$, and τ is a non-negative real number. Let S denote the set of all these states. We call S the modified set of states. Then for the process to be in the modified state (i, τ) at time t means that the process is in the elementary state E_i at time t and has been there for exactly the last τ units of time. It follows from this that two modified states (k, τ) and (j, λ) are the same if and only if $k = j$ and $\lambda = \tau$. We can represent S pictorially by the following diagram consisting of N half lines.



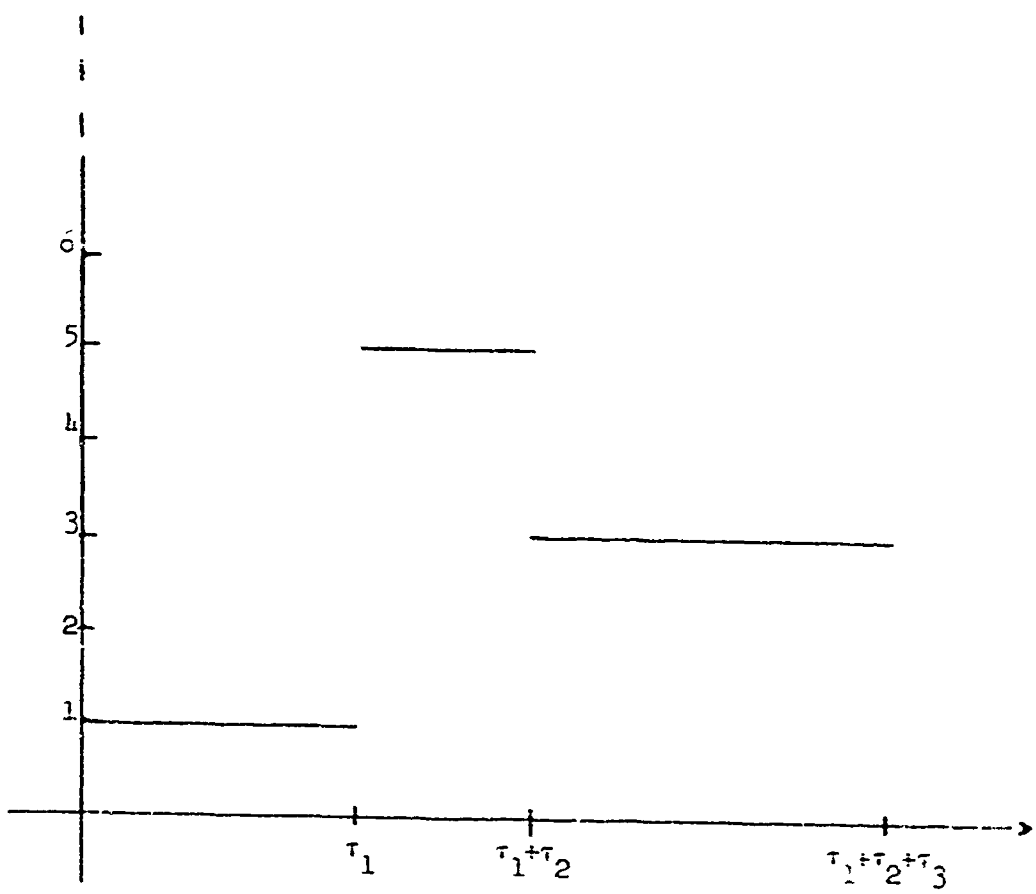
A typical sequence of transitions, starting from the elementary state E_1, at time t = 0 might look like



One may think of this process as operating as follows. There is an "epoch clock" which is initialized at time t = 0 upon the start of the process, say in state E_{i_o}. This "epoch clock" keeps advancing in t as the process continues. As the process transfers amongst the elementary states $\{E_i\}_{i=1}^N$, the "state

clock" is initialized at time $\tau = 0$ whenever an elementary state is entered. The "state clock" keeps track of the time spent, τ, in the elementary state E_i until transition, and is reset at $\tau = 0$ upon entering a new elementary state, E_j.

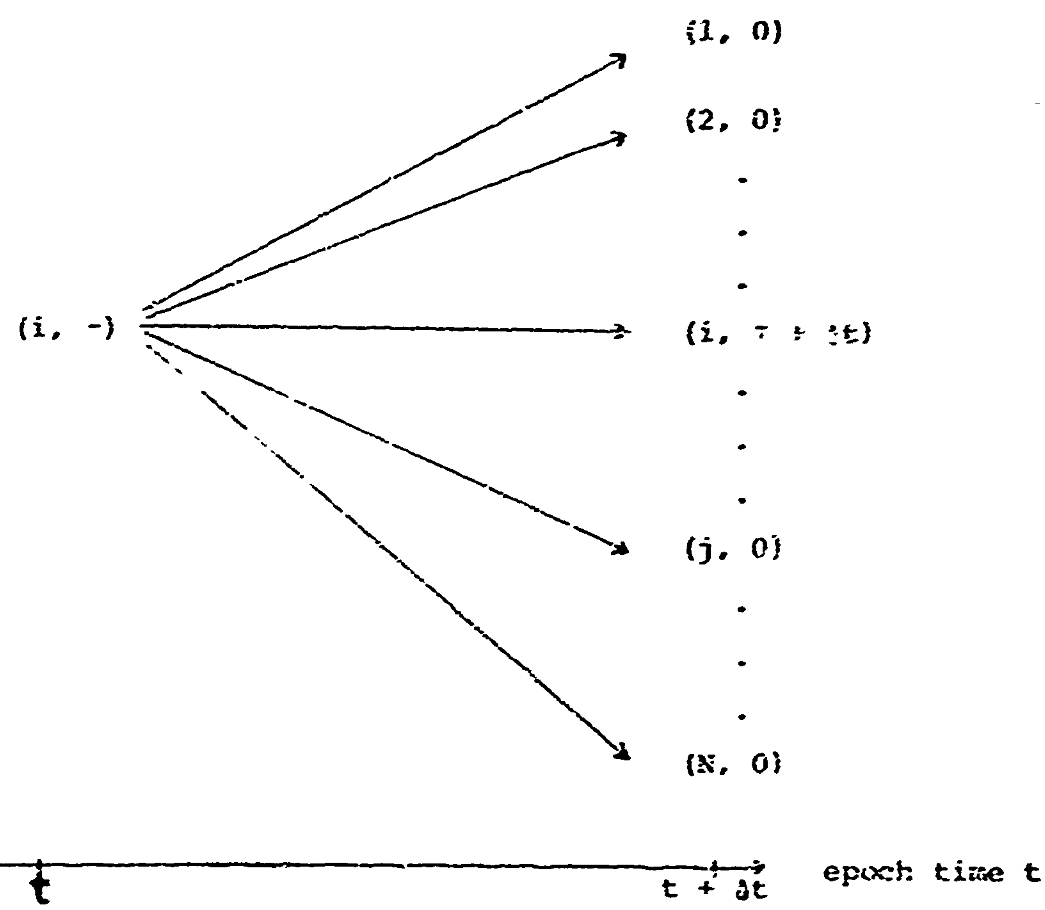
Thus, the sequence of transitions illustrated above can be represented in an "epoch clock" scale by



where τ_j represents the amount of time spent in the j^{th} elementary state, and j now refers to the order in which the elementary states are visited.

We observe now that our process with its modified states is Markovian. All the previous history needed to make a transition from the elementary state E_i having been there τ units of time is incorporated into the modified state (i, τ).

In general, given that the process is in (i, τ) at time t, the diagram below indicates the possible transitions the process may make by time $t + \Delta t$.



From equation (1), we see that the probability of transition from (i, τ) to $(j, 0)$ is given by $w_{ij}(\tau)\Delta\tau$. Furthermore, we assume that the probability of transition from (i, τ) to $(i, \tau + \Delta\tau)$ is given by

$$w_{ii}(\tau)\Delta\tau = 1 - \sum_{\substack{j=1 \\ j \neq i}}^{N} w_{ij}(\tau)\Delta\tau \tag{2}$$

THE GENERALIZED CHAPMAN-KOLMOGOROV EQUATIONS FOR THE SEMI-MARKOV PROCESS

In accordance with the definition of the set of modified states S, let $(X,T)_t$ be a 2-dimensional random variable, where X is discrete and T is continuous. In particular, X takes on the values 1, 2, ..., N and T is a non-negative real number. For a given epoch time $t > 0$, we wish to determine the probability density function for $(X,T)_t$. We denote it by $\phi_j(\tau,t)$ for $1 \leq j \leq N$, where

$$\phi_j(\tau, t)\Delta\tau = \mathrm{Prob}\left\{X = j,\ \tau \leq T < \tau+\Delta\tau \,\middle|\, t\right\}. \tag{3}$$

Now consider the possible (single) transitions during a time interval $[t, t + \Delta t)$ for the cases (a) $\tau > o$ and (b) $\tau = 0$

(a) $\tau > 0$: If at time $t + \Delta t$, $(X, T)_{t+\Delta t} = (j, \tau + \Delta t)$, then at time t we must have had that $(X, T)_t = (j, \tau)$ (for sufficiently small Δt). Thus we have

$$\phi_j(\tau + \Delta t, t + \Delta t) = \phi_j(\tau, t)\left[1 - \sum_{k \neq j} w_{jk}(\tau)\Delta t\right] + o(\Delta t)$$

which yields the partial differential equation

$$\frac{\partial \phi_j(\tau, t)}{\partial \tau} + \frac{\partial \phi_j(\tau, t)}{\partial t} = - w_j(\tau)\phi_j(\tau, t) \tag{4}$$

where

$$w_j(\tau) = \sum_{k \neq j} w_{jk}(\tau). \tag{5}$$

(b) $\tau = 0$: This is the case in which the process just entered the elementary state E_j during the time interval, $[t, t + \Delta t)$. So if at time t, $(X,T)_t = (i,s)$ for some i and s, then at time $t + \Delta t$, $(X,T)_{t+\Delta t} = (j,0)$. Thus the probability that $(X,T)_t$ has the value $(j,0)$ during the time interval $[t, t + \Delta t)$ is given by

$$\phi_j(0,t)\Delta t + o(\Delta t).$$

On the other hand since the transition to $(j,0)$ can arise from any $(i,s) \neq (j,0)$, we have for the probability that $(X,T)_t = (j,0)$ during $[t, t + \Delta t)$ the relation

$$\int_0^{t+\Delta t} \sum_{i \neq j} \phi_i(s,t) w_{ij}(s) \Delta t ds.$$

Equating the last two expressions and letting $\Delta t \to 0$ we get for $1 \leq j \leq N$ and $t > 0$

$$\phi_j(0,t) = \sum_{i \neq j} \int_0^t \phi_i(s,t) w_{ij}(s) ds. \tag{6}$$

Actually this equation is incomplete. It provides no contribution for an initial distribution (at $t = 0$) for the process. So let $\emptyset_j$ denote the probability that the process is initially in the elementary state E_j; that is,

$$\emptyset_j = \text{Prob}\ \{X = j \mid t = 0\} .$$

It follows that $\sum_{j=1}^{N} \emptyset_j = 1$. Now let $\eta(t)$ be the function

$$\eta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

The complete expression for $\emptyset_j(0,t)$ becomes

$$\emptyset_j(0,t) = \sum_{i \neq j} \int_0^t \emptyset_i(s,t) w_{ij}(s) ds + \emptyset_j \eta(t) \tag{7}$$

for all $t \geq 0$.

It can be shown that the solution to equation (4) is

$$\emptyset_j(\tau,t) = H_j(t - \tau) e^{-W_j(\tau)} \tag{8}$$

for $0 < \tau \le t$, where

$$W_j(\tau) = \int_0^\tau w_j(s)ds \tag{9}$$

and H_j is an arbitrary differentiable function. We choose H_j so that it is consistant with equation (6). In particular we define

$$H_j(t) = \emptyset_j(0,t). \tag{10}$$

We can interpret $H_j(t)$ as the probability density that the process just enters E_j at time t.

THE WAITING TIME DISTRIBUTIONS

Let us now examine the term $e^{-W_j(\tau)}$ in equation (8). If T_j is a random variable which denotes the time the process waits in state E_j before leaving for some other state, we define

$$P_j(\tau) = \text{Prob}\left\{T_j < \tau\right\}.$$

We assume that $P_j(0) = 0$ and that the process must eventually leave state E_j. Hence $\lim_{t\to\infty} P_j(t) = 1$. Then since $w_j(\tau)\Delta\tau$ is the probability that the process is in E_j during the interval $[\tau, \tau + \Delta\tau)$, conditioned on being in E_j during $[0,\tau)$, we find

$$w_j(\tau)\Delta\tau = \frac{\text{Prob}\left\{\tau \le T_j < \tau + \Delta\tau\right\}}{\text{Prob}\left\{T_j \ge \tau\right\}}$$

$$= \frac{P_j(\tau + \Delta\tau) - P_j(\tau)}{1 - P_j(\tau)} .$$

If the distribution function $P_j(\tau)$ is continuous, then we get upon letting $\Delta\tau \to 0$,

$$w_j(\tau)d\tau = \frac{dP_j(\tau)}{1 - P_j(\tau)} \tag{11}$$

Solving equation (11) for $P_j(\tau)$ we have

$$P_j(\tau) = 1 - e^{-W_j(\tau)} . \tag{12}$$

where $W_j(\tau)$ is given by equation (9).

Thus $e^{-W_j(\tau)}$ represents the probability that the process upon entering state E_j remains there for at least τ units of time. Furthermore,

$$P_j'(\tau) = w_j(\tau)e^{-W_j(\tau)} \tag{13}$$

is the probability density for the random variable T_j (where the prime denotes differentiation).

Upon substitution (of equations (8) and (10)) into equation (7) we get

$$H_j(t) = \sum_{i\neq j} \int_0^t H_i(t - s)w_{ij}(s)e^{-W_i(s)} ds + \phi_j r(t) \tag{14}$$

for $1 \leq j \leq N$. Setting

$$q_{ij}(s) = w_{ij}(s)e^{-W_i(s)} \tag{15}$$

for $1 \leq i, j \leq N$, $i \neq j$ and substituting into equation (14) we have

$$H_j(t) = \sum_{i\neq j} \int_0^t H_i(t - s)q_{ij}(s)ds + \phi_j r(t). \tag{16}$$

Now consider the term $q_{ij}(s)$. Then

$$q_{ij}(s)\Delta s = w_{ij}(s)\,[1 - P_i(s)]\;\Delta s$$

$$= \text{Prob}\left\{\begin{array}{l} X_s = i,\ X_{s+\Delta s} = j \\ \text{and } X_r = i \text{ or } j \\ \text{for } s < r < s+\Delta s \end{array}\middle|\ \begin{array}{l} X_v = i \\ 0 \le v < s \end{array}\right\} \cdot \text{Prob}\left\{\begin{array}{l} X_v = i \\ 0 \le v < s \end{array}\right\}$$

$$= \text{Prob}\left\{\begin{array}{l} X_s = i,\ X_{s+\Delta s} = j \\ \text{and } X_r = i \text{ or } j \\ \text{for } s < r < s+\Delta s \end{array}\right\}$$

= probability that the process makes a direct transition from E_i to E_j during the interval $[s, s + \Delta s)$. So

$$q_{ij} = \int_0^\infty q_{ij}(s)ds \qquad (17)$$

is the probability that the process, if in E_i at time $s = 0$, eventually jumps to E_j. Since we require that the process must eventually leave E_i for some other state, it follows that $q_{ii} = 0$. We note that

$$\sum_{j\neq i} q_{ij} = \sum_{j\neq i} \int_0^\infty w_{ij}(x)e^{-W_i(x)}dx$$

$$= \int_0^\infty w_i(x)e^{-W_i(x)}dx$$

$$= \int_0^\infty dP_i(x)$$

$$= 1 .$$

If we denote by Q the matrix of transition probabilities q_{ij}, for $1 \leq i,\ j \leq N$, then the set of elementary states $\{E_i\}_{i=1}^N$ along with the stochastic matrix Q define a Markov chain. Henceforth we assume that this is an irreducible chain.

Now suppose T_{ij} denotes the random variable representing the time the process waits in E_i before leaving for $E_j (j \neq i)$. (Of course this assumes prior knowledge of a transition to E_j.) Then the conditional probability density for T_{ij}, conditioned on the process making a transition from E_i to E_j is given by

$$t_{ij}(x) = \frac{1}{q_{ij}} q_{ij}(x). \tag{18}$$

We make the following assumption concerning this density: the mean time $\bar{t}_{ij}$ defined by

$$\bar{t}_{ij} = \int_0^\infty x\, t_{ij}(x)dx \tag{19}$$

is finite.

The following relationship is evident from the waiting time densities of equation (18). If $\bar{t}_i$ denotes the mean waiting time in state E_i, then since by definition

$$\bar{t}_i = \int_0^\infty x dP_i(x)$$

and since

$$dP_i(x) = \sum_{j \neq i} w_{ij}(x) e^{-W_i(x)} dx$$

we have

$$\begin{aligned} \bar{t}_i &= \int_0^\infty x \sum_{j \neq i} w_{ij}(x) e^{-W_i(x)} dx \\ &= \sum_{j \neq i} \int_0^\infty x q_{ij}(x) dx \\ &= \sum_{j \neq i} q_{ij} \bar{t}_{ij} < \infty. \end{aligned} \tag{20}$$

We note here that the transition probabilities q_{ij} and the conditional waiting time distributions $t_{ij}(x)$ are exactly the quantities postulated in references (c), (d), and (e). It follows from this that the transition mechanism given by equation (1) is equivalent to that given in the references just cited. For a transition matrix $Q = (q_{ij})$ and conditional waiting time densities $t_{ij}(x)$, there corresponds a unique $w_{ij}(x)$. In fact

$$w_{ij}(x) = \frac{q_{ij}t_{ij}(x)}{1 - P_i(x)},$$

Conversely, the transition function $w_{ij}(x)$ uniquely determines the quantities q_{ij} and $t_{ij}(x)$ as we have seen above.

SOLUTION OF THE EQUATIONS FOR THE SEMI-MARKOV PROCESS

Equations (8) and (16) are sufficient to determine each $\phi_j(\tau, t)$. Standard theorems of differential equations guarantee the existence and uniqueness of $H_j(t)$. One of these theorems, the Picard theorem of successive approximations, can be applied to develop an algorithm for the computation of $H_j(t)$.*

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* Besides formulating an algorithm for solution, we introduce the approximation procedure at this point in order to display equation (23)(to follow). This is needed in a later section.

To facilitate this, we put equation (16) in vector form.

Set

$$H(t) = \left[H_1(t),\ H_2(t),\ \ldots,\ H_N(t)\right].$$

Let

$$Q(t) = \begin{bmatrix} 0 & q_{12}(t) & q_{13}(t) \ - - - & q_{1N}(t) \\ q_{21}(t) & 0 & q_{23}(t) \ - - - & q_{2N}(t) \\ \vdots & \vdots & \vdots & \vdots \\ q_{N1}(t) & q_{N2}(t) & q_{N3}(t) \ - - - & 0 \end{bmatrix}$$

and

$$\phi = \left[\phi_1,\ \phi_2,\ \ldots,\ \phi_N\right].$$

Then

$$H(t) = \int_0^t H(t - s)Q(s)ds + \eta(t)\phi. \qquad (21)$$

We proceed towards the solution as follows. Let $[0, t]$ be the interval over which we want to determine H. Let

$$P = \left\{t_0 = 0, t_1, t_2, \ldots, t_{n-1}, t_n = t\right\}$$

be a partition of $(0, t]$ with norm Δ. What we shall do is to generate a sequence of functions $\{H^{(k)}\}$, each $H^{(k)}$ being defined on the points of the partition P. For sufficiently small Δ as $k \to \infty$, the sequence $\{H^{(k)}\}$ will converge uniformly to H on $(0, t]$. Initially let $H^{(0)}(t_j) = \emptyset$ for all $j = 0, 1, 2, \ldots, n$. Next set

$$H^{(1)}(t_j) = \int_0^{t_j} H^{(0)}(t_j - s)Q(s)ds + \eta(t_j)\emptyset.$$

Thus

$$H^{(1)}(t_j) = \emptyset \int_0^{t_j} Q(s)ds + \eta(t_j)\emptyset. \tag{22}$$

Similarly set

$$H^{(k+1)}(t_j) = \int_0^{t_j} H^{(k)}(t_j - s)Q(s)ds + \eta(t_j)\phi. \tag{23}$$

Thus from equation (23) if we know $Q(s)$ and $H^{(k)}(s)$ at all points of the partition P, $H^{(k+1)}$ is fully determined on P. The boundedness of $Q(s)$ insures that $H^{(k)}$ converges (uniquely) to H.

STEADY STATE ANALYSIS

Define

$$\Psi_j(t) = \int_0^t \phi_j(\tau, t)d\tau + \eta(t)\phi_j. \tag{24}$$

In appendix B we verify that $\phi_j(\tau, t)$ is an honest probability density for the joint random variables $(X, T)_t$. Thus $\Psi_j(t)$ is the marginal distribution (as a function of t) for the random variable X. In particular

$$\Psi_j(t) = \text{Prob}\left\{X_t = j\right\}.$$

We wish to study the behavior of $\Psi_j(t)$ at equilibrium. That is, we are interested in the limit

$$\Psi_j = \lim_{t\to\infty} \Psi_j(t)$$

if it exists. Since

$$\psi_j(t) = \int_0^t H_j(t-\tau)e^{-W_j(\tau)}d\tau + r(t)\emptyset_j \tag{25}$$

we note that this is a convolution integral and proceed to apply the method of Laplace transforms. Denoting the Laplace transform of a function $f(x)$ by $f^*(s)$, where

$$f^*(s) = \int_0^\infty e^{-sx}f(x)dx$$

for real valued s, $s \geq 0$, and using the convolution theorem of Laplace transforms, equation (25) yields

$$\psi_j^*(s) = H_j^*(s)E_j^*(s), \tag{26}$$

where $E_j^*(s)$ is the Laplace transform of $e^{-W_j(\tau)}$.

We now employ the well known Tauberian limit theorem which says

$$\lim_{x\to\infty} f(x) = \lim_{s\to 0^+} sf^*(s)$$

whenever one of the limits exists. Applying this to equation (26) we have

$$\Psi_j = \lim_{s\to 0^+} s\psi_j^*(s) = \left[\lim_{s\to 0^+} sH_j^*(s)\right]\left[\lim_{s\to 0^+} E_j^*(s)\right]. \tag{27}$$

In order to compute $\lim_{s\to 0^+} sH_j^*(s)$ we go back to equation (21). Taking Laplace transforms there (of the vector quantities) we get

$$H^*(s) = H^*(s)Q^*(s) \tag{28}$$

Multiplying by s and taking limits as $s \to 0^+$ yields

$$\left[\lim_{s\to 0^+} sH^*(s)\right] = \left[\lim_{s\to 0^+} sH^*(s)\right]\left[\lim_{s\to 0^+} Q^*(s)\right] \tag{29}$$

Since each element of $Q^*(s)$ is some $q_{ij}^*(s)$, where

$$q_{ij}^*(s) = \int_0^\infty e^{-st} q_{ij}(x)dx;$$

it follows that

$$\lim_{s\to 0^+} q^*(s) = \int_0^\infty q_{ij}(x)dx = q_{ij}$$

from equation (17). Thus

$$\lim_{s\to 0^+} Q^*(s) = Q.$$

From the theory of Markov chains we know that there exists a vector

$$\pi = \left[\pi_1, \pi_2, \ldots, \pi_N\right]$$

unique up to some positive factor such that

$$\pi = \pi Q$$

The vector π is just the limiting or steady state probabilities for the Markov chain defined by the stochastic matrix Q. Thus we must have

$$\pi = \lim_{s \to 0} sH^{*}(s) \qquad (30)$$

up to some positive factor.

It now remains to evaluate $\lim_{s \to 0^+} E^{*}(s)$. Since

$$e^{-W_j(t)} = 1 - P_j(t),$$

Then

$$E_j^{*}(s) = \frac{1}{s} - P_j^{*}(s).$$

But

$$\int_0^\infty e^{-st} dP_j(t) = sP_j^*(s) - P_j(0).$$

Substituting this in the expression for $F_j^*(s)$ (noting that $P_j(0) = 0$) we get

$$E_j^*(s) = \frac{1}{s}\left[1 - \int_0^\infty e^{-st} dP_j(t)\right]. \tag{31}$$

But the limit here as $s \to 0^+$ is indeterminate, so we use L' Hospital's rule. This gives

$$\lim_{s \to 0^+} E_j^*(s) = - \frac{d}{ds}\int_0^\infty e^{-st} dP_j(t) \Bigg|_{s = 0}$$

$$= \int_0^\infty te^{-st} dP_j(t) \Bigg|_{s = 0}$$

$$= \int_0^\infty t dP_j(t)$$

$$= \bar{t}_j \tag{32}$$

Thus the limit yields the mean time spent in state E_j prior to transition. If $\bar{t}_j$ and π_j are placed in equation (27) we are left with $Y_j = \pi_j \bar{t}_j$ up to a constant

factor. Normalizing we have

$$\Psi_i = \frac{\pi_j \bar{t}_j}{\sum_{j=1}^{N} \pi_j \bar{t}_j} \tag{33}$$

Thus the steady state probability of being in E_j is essentially the probability π_j of being in E_j of the imbedded Markov chain given by Q but weighted by the expected waiting times in each state. Furthermore we note the independence of Ψ_j with respect to the initial condition $\emptyset$. It is also evident that Ψ_j is independent of any of the specific waiting time distributions. We need only know the mean waiting time $\bar{t}_i$.

We can also calculate the steady state value of $\emptyset_j(\tau,t)$, that is,

$$\emptyset_j(\tau) = \lim_{t\to\infty} \emptyset_j(\tau, t).$$

By equation (8) we get

$$\phi_j(\tau) = \lim_{t\to\infty} H_j(t - \tau)e^{-W_j(\tau)}$$

$$= \pi_j e^{-W_j(\tau)}$$

(up to a constant factor). Upon normalizing we get

$$\phi_j(\tau) = \frac{\pi_j e^{-W_j(\tau)}}{\sum_{j=1}^{N} \pi_j \bar{t}_j} . \tag{34}$$

Thus at steady state the probability of being in E_j for exactly the last τ units of time is just the equilibrium probability π_j of being in E_j of the imbedded Markov chain times the probability of remaining in E_j for at least τ units of time.

We note here that since

$$\int_0^\infty e^{-W_j(\tau)} d\tau = \bar{t}_j,$$

then

$$\Psi_j = \int_0^{\infty} \phi_j(\tau)d\tau$$

as we should expect.

FIRST PASSAGE AND RECURRENCE DISTRIBUTIONS

Let $r_{ij}(t)$ be the probability density function for first passage measured from the moment the process enters E_i until it first enters E_j. In particular then, $r_{ii}(t)$ is the probability density for the recurrence time of state E_i. We will show that

$$r_{ij}(t) = \sum_{k \neq j} q_{ik} \int_0^{\infty} t_{ik}(s) r_{kj}(t - s)ds + q_{ij}(t). \qquad (35)$$

Indeed, there are two ways that the process can first arrive at E_j at time t, having entered E_i at t = 0. Either the process remained in E_i for a time s, and left for E_k whence it took t - s units of time to first get to E_j or the process remained in E_i for the whole time t at which point it jumped to E_j. The integral in equation (35) reflects the former possibility. Since the conditional waiting time in E_i before transfer to E_k is independent of the first passage time from E_k to E_j the integral in equation (35)

is the probability density function for the time t of first passage via the route E_i to E_k to E_j. Since we can make all of these transitions (through E_k) we must weight them according to their eventual likelihood of occurrence, namely the q_{ij}'s. The latter possibility for first passage to E_j at time t is represented by the unconditional probability $q_{ij}(t)$.

The expected first passage time $\bar{r}_{ij}$ can be shown to satisfy the following equation for all $1 \le i, j \le N$.

$$\bar{r}_{ij} = \sum_{k \neq j} q_{ik}(\bar{t}_{ik} + \bar{r}_{kj}) + q_{ij}\,\bar{t}_{ij} \tag{36}$$

An exact derivation of equation (36) is presented in appendix A. At this point we give a heuristic proof of equation (36).

Suppose the process enters E_i at time t = 0. At this point a choice is made as to what elementary state the process jumps to next. Suppose the successor is E_k where $k \neq j$. Then the process waits in E_i for an expected time $\bar{t}_{ik}$ before going to E_k. Since the waiting time in E_i is independent of the first passage time from E_k to E_j, the expected time from entry to E_i until first entry to E_j is $\bar{t}_{ik} + \bar{r}_{kj}$. But the choice of E_k is determined by the probabilities q_{ik}. So we weight the times $\bar{t}_{ik} + \bar{r}_{kj}$

accordingly as in the first term on the r.h.s. of equation (38). The last term of equation (36) results from the choice to wait in E_i and go to E_j directly without passing through any intervening elementary states E_k. This waiting time, $\bar{t}_{ij}$ is also weighed accordingly.

Substituting $\bar{t}_i$ for $\sum_{k \neq i} q_{ij}\bar{t}_{ik}$, we have from equation (36) that

$$\bar{r}_{ij} = \sum_{k \neq j} q_{ik}\bar{r}_{kj} + \bar{t}_i. \tag{37}$$

We are primarily interested in obtaining the expected recurrence time $\bar{r}_{ii}$ for $1 \le i \le N$. This is simplified by putting equation (37) in matrix notation. Letting R denote the matrix of terms $\bar{r}_{ij}$ and T the diagonal matrix of the $\bar{t}_i$'s, we get

$$R = Q(R - R_D) + T$$

where R_D is obtained from R by replacing the off diagonal elements by zero. Letting $R_O = R - R_D$ we get

$$R_D = (Q - I)R_O + T \tag{38}$$

Now multiply both sides of equation (38) by π, the vector of steady state probabilities for the imbedded Markov process, and we get $\pi R_D = \pi T$. Then solving for $\bar{r}_{ii}$ we have

$$\bar{r}_{ii} = \frac{1}{\pi_i} \sum_{k=1}^{N} \pi_k \bar{t}_k. \tag{39}$$

APPENDIX A

DERIVATION OF THE EXPECTED FIRST PASSAGE TIMES

By definition,

$$\bar{r}_{ij} = \int_{-\infty}^{\infty} t r_{ij}(t) dt.$$

Then from equation (37)

$$\bar{r}_{ij} = \int_{-\infty}^{\infty} t \left[\sum_{k \neq j} q_{ik} \int_{-\infty}^{\infty} t_{ik}(s) r_{kj}(t - s) ds \right] dt + \int_{-\infty}^{\infty} t q_{ij}(t) dt$$

$$= \sum_{k \neq j} q_{ik} \int_{-\infty}^{\infty} t_{ik}(s) \left[\int_{-\infty}^{\infty} t r_{kj}(t - s) dt \right] ds + q_{ij} \bar{t}_{ij}$$

$$= \sum_{k \neq j} q_{ij} \int_{-\infty}^{\infty} t_{ik}(s) \left[\int_{-\infty}^{\infty} (s + x) r_{kj}(x) dx \right] ds + q_{ij} \bar{t}_{ij}$$

(where we have made the change of variables $x = t - s$)

$$= \sum_{k \neq j} q_{ik} \int_{-\infty}^{\infty} t_k(s) \left[s + \bar{r}_{kj} \right] ds + q_{ij} \bar{t}_{ij}$$

$$= \sum_{k \neq j} q_{ik} \int_{-\infty}^{\infty} \left[s t_{ik}(s) + \bar{r}_{kj} t_{ik}(s) \right] ds + q_{ij} \bar{t}_{ij}$$

$$= \sum_{k \neq j} q_{ik} \left[\bar{t}_{ik} + \bar{r}_{kj} \right] + q_{ij} \bar{t}_{ij}$$

using equation (19) to evaluate $\bar{t}_{ik}$.

APPENDIX B

PROOF THAT ϕ_j (τ, t) IS A PROBABILITY DENSITY

In order to show that $\phi_j(\tau, t)$ is an "honest" joint probability density for the random variable $(X, T)_t$, we must prove

$$\text{(i)} \quad \phi_j(\tau, t) \geq 0$$

for all $j = 1, 2, \ldots, N$ and all τ, $0 \leq \tau \leq t$, and

$$\text{(ii)} \quad \sum_{j=1}^{N} \int_0^t \phi_j(\tau, t)\, d\tau + \eta(t) \sum_{j=1}^{N} \phi_j = 1.$$

To prove (i) we consider equation (23) for any fixed $t = t_j$. We first show that the vector $H(t) \geq 0$, i.e., has all non-negative components. Now by definition of the matrix $Q(s)$, each entry is non-negative, hence $Q(s) \geq 0$. Since

$$H^{(0)}(t) = \phi \geq 0$$

then by equation (22), $H^{(1)}(t) \geq 0$. Similarly from equation (23) we see that $H^{(k)}(t) \geq 0$ implies that $H^{(k+1)}(t) \geq 0$.

It then follows from the uniform convergence of $H^{(k)}$ to H on $(0,t]$ that $H(t) \geq 0$. Since each $H_j(t) \geq 0$ then

$$\phi_j(\tau, t) = H_j(t - \tau)e^{-W_j(\tau)} \geq 0.$$

For (ii) we define the function

$$\Psi_j(t) = \int_0^t \phi_j(\tau,t)d\tau + \eta(t)\phi_j.$$

Then substitution for $\phi_j(\tau,t)$ from equation (8) and (10), we get

$$\Psi_j(t) = \int_0^t e^{-W_j(t - v)} \left\{ \sum_{i \neq j} \int_0^v \phi_i(s, v)w_{ij}(s)ds + \delta(v)\phi_j \right\} dv + \eta(t)\phi_j$$

$$= \int_0^t e^{-W_j(t - v)} \left\{ \sum_{i \neq j} \int_0^v \phi_i(s, v)w_{ij}(s)ds \right\} dv + \eta(t)\phi_j.$$

Taking derivatives with respect to t yields

$$\Psi_j'(t) = - \int_0^t w_j(t - v) e^{-W_j(t - v)} \left\{ \sum_{i \neq j} \int_0^v \phi_i(s, v) w_{ij}(s) ds \right\} dv$$

$$+ \sum_{i \neq j} \int_0^t \phi_i(s, t) w_{ij}(s) ds.$$

We can introduce the term $\eta(v)\phi_j$ into the first integral (with respect to v) without changing its value. Then

$$\Psi_j'(t) = - \int_0^t w_j(t - v) \, e^{-W_j(t-v)} \left\{ \sum_{i \neq j} \int_0^v \phi_i(s, v) w_{ij}(s) ds + \eta(v)\phi_j \right\} dv$$

$$+ \sum_{i \neq j} \int_0^t \phi_i(s, \varepsilon) w_{ij}(s) ds.$$

If we replace the term in brackets by $H_j(v)$ we are left with

$$\Psi_j'(t) = -\int_0^t w_j(t - v)e^{-W_j(t-v)} H_j(v)dv + \sum_{i\neq j} \int_0^t \phi_i(s, t)w_{ij}(s)ds$$

$$= -\int_0^t w_j(t - v)\phi_j(t - v, t)dv + \sum_{i\neq j} \int_0^t \phi_i(s, t)w_{ij}(s)ds.$$

Making a change of variables in the first integral we get

$$\Psi_j'(t) = -\int_0^t \phi_j(s, t)w_j(s)ds + \sum_{i\neq j} \int_0^t \phi_i(s, t)w_{ij}(s)ds.$$

Summing over all states j we have

$$\sum_{j=1}^{N} \Psi_j'(t) = -\sum_{j=1}^{N} \int_0^t \phi_j(s, t) w_j(s) ds + \sum_{j=1}^{N} \sum_{i \neq j} \int_0^t \phi_i(s, t) w_{ij}(s) ds$$

$$= -\sum_{j=1}^{N} \int_0^t \phi_j(s, t) \sum_{k \neq j} w_{jk}(s) ds + \sum_{j=1}^{N} \sum_{i \neq j} \int_0^t \phi_i(s, t) w_{ij}(s) ds$$

$$= -\sum_{j=1}^{N} \sum_{k \neq j} \int_0^t \phi_j(s, t) w_{jk}(s) ds + \sum_{j=1}^{N} \sum_{i \neq j} \int_0^t \phi_i(s, t) w_{ij}(s) ds$$

$$= 0.$$

Observe that

$$\sum_{j=1}^{N} \Psi_j(0) = \sum_{j=1}^{N} \phi_j = 1,$$

hence

$$\sum_{j=1}^{N} \Psi_j(t) = 1$$

for all $t \geq 0$. But this is just the statement (ii) we wished to prove.

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